

THE UNIVERSAL DEFORMATION OF THE WITT RING SCHEME

CHRISTOPHER DENINGER AND YOUNG-TAK OH

1. INTRODUCTION

Witt vectors play an important role in several branches of mathematics. Combinatorial considerations led to the study of certain q -deformations over $\operatorname{spec} \mathbb{Z}[q]$ of the big Witt vector scheme W over $\operatorname{spec} \mathbb{Z}$, c.f. [O2]. In the present note we consider another and simpler q -deformation of W to a non-unital ring scheme $W^{(q)}$. The main result asserts that $W^{(q)}$ enhanced by a Frobenius lift, Verschiebung and the choice of a coordinate for the first component is the universal deformation over reduced bases of W with the corresponding structures. It follows that the triple $(W, \text{Frobenius lift, Verschiebung})$ has no deformations within unital ring-schemes and a one-parameter deformation in the non-unital category with parameter “space” $\mathbb{G}_a/\mathbb{G}_m$ over $\mathbb{Z}[q]$. The q -deformation of [O2] turns out to be isomorphic to $W^{(q)}$.

Over non-reduced bases the deformation theory of the triple $(W, \text{Frobenius lift, Verschiebung})$ is richer and Theorem 3 in section 4 allows to determine it in principle. The theorem also allows a simple proof of the decomposition theorem $W_{S \cdot T} = W_S \circ W_T$ for coprime divisor stable sets S and T due to Auer, c.f. [A]. At the end of section 5 we discuss the deformations of W for integer values of q studied in [L] and [O1].

Throughout the paper we work with Witt vectors for divisor stable sets. Since in general Frobenius and Verschiebung are not endomorphisms of those, one needs to work with projective systems of rings indexed by divisor stable subsets. As our main technical tool we prove a Cartier–Dieudonné theorem for them. A direct proof is given in section 3. A more conceptual proof is also possible using the theory of Witt vectors for certain inductive systems of rings which is developed in the appendix. This theory is a very natural generalization of Witt vector theory for individual rings and may be of independent interest.

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2. q -DEFORMED WITT VECTORS

Convention: In this note all rings and algebras will be commutative and associative but not always unital. As usual “non-unital” means “not necessarily unital”.

A non-empty subset S of the natural numbers $\mathbb{N} = \{1, 2, \dots\}$ is called *divisor stable* if $n \in S$ and $d \mid n$ imply that $d \in S$. In particular $1 \in S$. For $n \in S$ the sets $S/n = \{\nu \in S \mid \nu n \in S\} \subset S$ and $S(n) = \{\nu \in S \mid n \nmid \nu\} \subset S$ are again divisor stable. We assume that the reader is familiar with the rings of S -Witt vectors $W_S(A)$ defined for all commutative rings A . The ring $W_S(A)$ is unital if and only if A is unital. Set theoretically $W_S(A) = A^S$ and hence W_S is a (unital) ring-scheme over $\text{spec } \mathbb{Z}$ whose underlying scheme is $\mathbb{A}^S = \text{spec } \mathbb{Z}[S]$. Here $\mathbb{Z}[S] = \mathbb{Z}[t_n \mid n \in S]$ is the free commutative unital algebra on the set S or in other words the polynomial algebra in indeterminates t_n for $n \in S$. The values of co-addition and co-multiplication

$$\Delta_+, \Delta \bullet : \mathbb{Z}[S] \longrightarrow \mathbb{Z}[S] \otimes_{\mathbb{Z}} \mathbb{Z}[S] = \mathbb{Z}[x_n; y_m \mid n, m \in S]$$

on the generators $n \in S$ are the Witt polynomials Σ_n and Π_n defining addition and multiplication in $W_S(A) = A^S$ i.e.:

$$\Sigma_n = \Delta_+(t_n), \Pi_n = \Delta \bullet(t_n) \quad \text{in } \mathbb{Z}[x_\nu; y_\mu \mid \nu \mid n, \mu \mid n] .$$

Here $x_\nu = t_\nu \otimes 1$ and $y_\mu = 1 \otimes t_\mu$. For every divisor stable subset S' of S there is a natural projection morphism $\pi : W_S \rightarrow W_{S'}$ of ring-schemes. There is a morphism $s : W_{S'} \rightarrow W_S$ of schemes such that $\pi \circ s = \text{id}$. Equivalently $\pi : W_S(A) \rightarrow W_{S'}(A)$ is surjective for all rings A . For $n \in S$ Verschiebung $V_n : W_{S/n} \rightarrow W_S$ is an additive morphism which fits into an exact sequence

$$0 \longrightarrow W_{S/n} \xrightarrow{V_n} W_S \xrightarrow{\pi} W_{S(n)} \longrightarrow 0 .$$

Since $W_{\{1\}} = \mathbb{G}_a$ additively, it follows that for finite S the underlying additive group scheme of W_S is an iterated extension of \mathbb{G}_a 's. In particular $W_S \otimes \mathbb{Q}$ is a commutative unipotent group scheme over $\text{spec } \mathbb{Q}$ and hence the logarithm provides an isomorphism

$$\log : W_S \otimes \mathbb{Q} \xrightarrow{\sim} T_0 W_S \otimes \mathbb{Q}$$

of additive group schemes over $\text{spec } \mathbb{Q}$ c.f. [DG] IV, §2 n° 4 Proposition 4.1 (iii). Here $T_0 W_S$ is the tangent group scheme to W_S along the zero section. It is canonically identified with \mathbb{A}^S with componentwise addition. Consider the morphism $\underline{S} : \mathbb{A}^S \rightarrow \mathbb{A}^S$ which on A -valued points maps $(x_d)_{d \in S}$ to $(dx_d)_{d \in S}$. Then the additive isomorphism

$$\log_S = \underline{S} \circ \log : W_S \otimes \mathbb{Q} \xrightarrow{\sim} T_0 W_S \otimes \mathbb{Q}$$

is induced by a unique additive morphism

$$\log_S : W_S \longrightarrow T_0 W_S \cong \mathbb{A}^S .$$

This is the so-called ghost map. Explicitly, we have

$$\log_S((a_d)_{d \in S}) = \left(\sum_{d|n} da_d^{n/d} \right)_{n \in S}$$

on A -valued points. The preceding assertions also hold for arbitrary S as one sees by taking the projective limit over the finite divisor-stable subsets S_0 of S . In the standard approach to Witt vector theory, the ghost map is used to define the ring-structure. A new approach giving the ring-structure directly was introduced in [CD]. The morphism \log_S is a morphism of (unital) ring-schemes if the ring-scheme structure on $T_0W_S \cong \mathbb{A}^S$ is defined componentwise and \mathbb{A}^1 carries the standard ring-structure. With the identifications $\Gamma(W_S, \mathcal{O}) = \mathbb{Z}[x_d \mid d \in S]$ and $\Gamma(T_0W_S, \mathcal{O}) = \mathbb{Z}[u_d \mid d \in S]$ the induced ring-homomorphism

$$\log_S^* : \Gamma(T_0W_S, \mathcal{O}) \longrightarrow \Gamma(W_S, \mathcal{O})$$

is determined by the formulas

$$\log_S^*(u_n) = \sum_{d|n} dx_d^{n/d} \quad \text{for } n \in S.$$

These are also the components of the formal logarithm with Jacobian \underline{S} for the formal $|S|$ -dimensional group-law determined by the polynomials Σ_n for $n \in S$.

Remark. The map $\underline{S} : A^S \rightarrow A^S$ on the product ring A^S is additive. Viewing the elements of A^S as formal power series in x over A with exponents in S we have $\underline{S} = x \frac{d}{dx}$. In this context recall that if additively $W(A)$ is viewed as the formal multiplicative group of power series over A the ghost map is given by $x \frac{d}{dx} \circ \log$.

We now recall the Frobenius and Verschiebung morphisms. For $n \in S$, the Frobenius morphism $F_n : W_S \rightarrow W_{S/n}$ is a morphism of ring-schemes. The following relations hold in an evident sense

$$V_n \circ V_m = V_{nm} : W_{S/nm} \rightarrow W_S, \quad F_n \circ F_m = F_{nm} : W_S \rightarrow W_{S/nm} \text{ if } nm \in S$$

$$F_n \circ V_n = n \text{ id} : W_{S/n} \rightarrow W_{S/n} \text{ for } n \in S$$

$$F_n \circ V_m = V_m \circ F_n : W_{S/m} \rightarrow W_{S/n} \text{ for } (n, m) = 1 \text{ and } nm \in S.$$

For a prime number $p \in S$, the Frobenius morphism F_p reduces mod p to the p -th power map: the following diagram commutes for all rings A

$$(1) \quad \begin{array}{ccc} W_S(A) & \xrightarrow{F_p} & W_{S/p}(A) \\ \downarrow & & \downarrow \\ W_S(A)/p & \xrightarrow{(\cdot)^p} & W_S(A)/p \xrightarrow{\pi} W_{S/p}(A)/p. \end{array}$$

Here the vertical maps are the reduction maps mod p . Of course the bottom line could also be replaced by

$$W_S(A)/p \xrightarrow{\pi} W_{S/p}(A)/p \xrightarrow{(\cdot)^p} W_{S/p}(A)/p.$$

Finally consider the functor P on rings defined by $P(A) = (A, \cdot)$. It is represented by the monoid scheme $P = \operatorname{spec} \mathbb{Z}[t]$ with co-multiplication $\Delta \bullet$ and co-unit ε_1 determined by $\Delta \bullet(t) = t \otimes t$ and $\varepsilon_1(t) = 1$. The Teichmüller map is the multiplicative morphism

$$\omega : P \longrightarrow W_S$$

which on A -valued points sends $a \in A = P(A)$ to $(a\delta_{d,1})_{d \in S}$ in $A^S = W_S(A)$. Here $\delta_{\nu,\mu} = 1$ if $\nu = \mu$ and $\delta_{\nu,\mu} = 0$ if $\nu \neq \mu$. For every commutative unital ring R , the whole situation can be base-changed to $\operatorname{spec} R$ and we then speak of Witt vector schemes over R etc.

We now describe two deformations of Witt vector theory which will later turn out to be isomorphic and universal in a suitable sense. The first one, $W^{(q)}$ is obtained by a simple modification of the usual Witt vector functor. The second one, \overline{W}^{1-q} was introduced in [O2] using a q -deformed ghost map.

Let A be a unital algebra over the polynomial ring $\mathbb{Z}[q]$. We denote by $A^{(q)}$ the ring with underlying additive group $(A, +)$ and twisted multiplication $x * y = qxy$. It is unital if and only if A is a $\mathbb{Z}[q, q^{-1}]$ -algebra. Setting $W_S^{(q)}(A) = W_S(A^{(q)})$ we obtain a commutative non-unital ring scheme $W_S^{(q)}$ over $\operatorname{spec} \mathbb{Z}[q]$ whose underlying scheme is \mathbb{A}^S . The base change of $W_S^{(q)}$ to $\operatorname{spec} \mathbb{Z}[q, q^{-1}]$ is a unital ring-scheme. By the Yoneda lemma the Frobenius and Verschiebung maps for $n \in S$

$$F_n : W_S(A^{(q)}) \longrightarrow W_{S/n}(A^{(q)}) \quad \text{and} \quad V_n : W_{S/n}(A^{(q)}) \longrightarrow W_S(A^{(q)})$$

come from morphisms $F_n : W_S^{(q)} \rightarrow W_{S/n}^{(q)}$ and $V_n : W_{S/n}^{(q)} \rightarrow W_S^{(q)}$. They have the same properties as the ones recalled above for the usual Witt vector schemes. In particular the commutative group schemes $W_S^{(q)}$ are unipotent over $\operatorname{spec} \mathbb{Q}[q]$ and hence there is the log-isomorphism

$$\log : W_S^{(q)} \otimes \mathbb{Q} \xrightarrow{\sim} T_0 W_S^{(q)} \otimes \mathbb{Q}$$

of additive group schemes over $\operatorname{spec} \mathbb{Q}[q]$. Here $T_0 W_S^{(q)}$ is the tangent group scheme to $W_S^{(q)}$ over $\operatorname{spec} \mathbb{Z}[q]$ along the zero section. As before $T_0 W_S^{(q)} \cong \mathbb{A}^S$ canonically and we have the morphism \underline{S} defined as before. The additive isomorphism

$$\log_S = \underline{S} \circ \log : W_S^{(q)} \otimes \mathbb{Q} \xrightarrow{\sim} T_0 W_S^{(q)} \otimes \mathbb{Q}$$

is induced by a unique additive morphism over $\mathbb{Z}[q]$

$$\log_S : W_S^{(q)} \longrightarrow T_0 W_S^{(q)} \cong \mathbb{A}^S.$$

On A -valued points the induced (ghost-)map

$$(2) \quad \log_S : A^S \longrightarrow (A^{(q)})^S$$

is given by the formula

$$\log_S((a_d)_{d \in S}) = \left(\sum_{d|n} dq^{\frac{n}{d}-1} a_d^{\frac{n}{d}} \right)_{n \in S}.$$

Note here that $a^{*n} = q^{n-1}a^n$ for $a \in A, n \geq 1$ by definition of the q -twisted multiplication in $A^{(q)}$. The map \log_S in (2) is a ring-homomorphism if on the right we view $(A^{(q)})^S$ as a ring under componentwise addition and multiplication. On the left we take the ring-structure on the set A^S coming from the identification $W_S^{(q)}(A) = A^S$ as sets. Via the identifications $\Gamma(W_S^{(q)}, \mathcal{O}) = \mathbb{Z}[q][x_d \mid d \in S]$ and $\Gamma(T_0 W_S^{(q)}, \mathcal{O}) = \mathbb{Z}[q][u_d \mid d \in S]$ the $\mathbb{Z}[q]$ -algebra homomorphism

$$\log_S^* : \Gamma(T_0 W_S^{(q)}, \mathcal{O}) \longrightarrow \Gamma(W_S^{(q)}, \mathcal{O})$$

is given by the formulas

$$\log_S^*(u_n) = \sum_{d|n} dq^{\frac{n}{d}-1} x_d^{d/n} = q^{-1} \sum_{d|n} d(qx_d)^{n/d}.$$

It follows that the universal polynomials for addition and multiplication and the Frobenius and Verschiebung morphisms are obtained from the usual ones by multiplying the variables by q and dividing the resulting polynomial by q . For $S = \{1, p\}$ for example, setting $a = (a_1, a_p), b = (b_1, b_p)$ we have:

$$\begin{aligned} \Sigma_1(a, b) &= a_1 + b_1 \\ \Sigma_p(a, b) &= a_p + b_p - q^{p-1} \sum_{\nu=1}^{p-1} p^{-1} \binom{p}{\nu} a_1^\nu b_1^{p-\nu} \\ \Pi_1(a, b) &= qa_1b_1 \\ \Pi_p(a, b) &= qpa_p b_p + q^p(a_1^p b_p + a_p b_1^p). \end{aligned}$$

Consider the functor $P^{(q)}$ on $\mathbb{Z}[q]$ -algebras defined by $P^{(q)}(A) = (A^{(q)}, *) = (A, *)$ where $x * y = qxy$ as above. It is represented by the semigroup scheme $P^{(q)} = \text{spec } \mathbb{Z}[q][t]$ over $\text{spec } \mathbb{Z}[q]$ with co-multiplication $\Delta \bullet$ determined by $\Delta \bullet(t) = qt \otimes t$. The base change to $\mathbb{Z}[q, q^{-1}]$ is a monoid scheme with co-unit $\varepsilon_1(t) = q^{-1}$. There is a unique morphism $\omega^{(q)} : P^{(q)} \rightarrow W_S^{(q)}$ of multiplicative semigroup schemes which on $\mathbb{Z}[q]$ -algebras A becomes the ordinary Teichmüller map

$$(A^{(q)}, \bullet) \longrightarrow W_S(A^{(q)}), \quad \omega^{(q)}(a) = (a\delta_{d,1})_{d \in S}.$$

Base changed to $\mathbb{Z}[q, q^{-1}]$ the map $\omega^{(q)}$ becomes a morphism of monoid schemes.

Remark In [L] and [O1] certain q -deformations of W over $\text{spec } \mathbb{Z}$ were studied for integer values of q . The ghost map is the same as above but the induced multiplication on the Witt vectors is different since the ghost side is viewed as the ring A^S and not as $(A^{(q)})^S$. The additive structure is the same though.

For every polynomial $g(q)$ in $\mathbb{Z}[q]$ a ring scheme $\overline{W}^{g(q)}$ over $\mathbb{Z}[q]$ was introduced in [O2]. The construction can be generalized to every divisor-stable subset S of \mathbb{N} using the same arguments. The most relevant case for us is \overline{W}_S^{1-q} . It is defined as follows: There is a unique functorial ring structure on A^S for all $\mathbb{Z}[q]$ -algebras A such that the following “ghost” map is a homomorphism of non-unital rings

$$\mathcal{G}_S : A^S \longrightarrow (A^{(q)})^S, \quad \mathcal{G}_S((a_n)_{n \in S}) = \left(\sum_{d|n} dq^{-1}(1 - (1 - q)^{n/d})a_d^{n/d} \right)_{n \in S}.$$

It follows from Propositions 6.1 and 6.2 of [O2] that there are unique Frobenius and Verschiebung morphisms $F_n : \overline{W}_S^{1-q} \rightarrow \overline{W}_{S/n}^{1-q}$ and $V_n : \overline{W}_{S/n}^{1-q} \rightarrow \overline{W}_S^{1-q}$ that correspond via the ghost maps to the maps

$$F_n((a_\nu)_{\nu \in S}) = (a_{n\nu})_{\nu \in S/n} \quad \text{and} \quad V_n((a_\nu)_{\nu \in S/n}) = (n\delta_{n|\nu}a_{\nu/n})_{\nu \in S}.$$

They satisfy the same relations as for the usual Witt vectors and in particular F_p reduces mod p to the p -th power morphism. The modified log-morphism $\log_S = \underline{S} \circ \log$ exists over $\mathbb{Z}[q]$, i.e. $\log_S : \overline{W}_S^{1-q} \rightarrow T_0 \overline{W}_S^{1-q}$ and on A -valued points it equals \mathcal{G}_S . Writing

$$\Gamma(\overline{W}_S^{1-q}, \mathcal{O}) = \mathbb{Z}[q][x_d \mid d \in S] \quad \text{and} \quad \Gamma(T_0 \overline{W}_S^{1-q}, \mathcal{O}) = \mathbb{Z}[q][u_d \mid d \in S]$$

we have

$$\log_S^*(u_n) = \sum_{d|n} dq^{-1}(1 - (1 - q)^{n/d})x_d^{n/d} \quad \text{for } n \in S.$$

The universal polynomials describing \overline{W}_S^{1-q} are more complicated than the ones for $W_S^{(q)}$. For $S = \{1, p\}$ for example they are the following:

$$\Sigma_1(a, b) = a_1 + b_1$$

$$\Sigma_p(a, b) = a_p + b_p - h(q) \sum_{\nu=1}^{p-1} p^{-1} \binom{p}{\nu} a_1^\nu b_1^{p-\nu}$$

$$\Pi_1(a, b) = qa_1b_1$$

$$\Pi_p(a, b) = qpa_p b_p + qh(q)(a_1^p b_p + b_1^p a_p) + qh(q)r(q)a_1^p b_1^p.$$

Here we have set

$$h(q) = q^{-1}(1 - (1 - q)^p) = 1 + (1 - q) + \dots + (1 - q)^{p-1}$$

and

$$r(q) = p^{-1}(h(q) - q^{p-1}) = p^{-1}q^{-1}(1 - q^p - (1 - q)^p) \in \mathbb{Z}[q].$$

For the integrality of $r(q)$ note that

$$1 - q^p - (1 - q)^p \equiv 1 - q^p - (1 - q^p) \pmod{p} \equiv 0 \pmod{p}$$

and

$$1 - q^p - (1 - q)^p \equiv 0 \pmod{q}.$$

For any polynomial $g(q) \in \mathbb{Z}[q]$ consider the ring homomorphism $\alpha : \mathbb{Z}[q] \rightarrow \mathbb{Z}[q]$ with $\alpha(q) = 1 - g(q)$ and set

$$\overline{W}_S^{g(q)} = \overline{W}_S^{1-q} \otimes_{\mathbb{Z}[q], \alpha} \mathbb{Z}[q] .$$

In [O2] the non-unital ring schemes $\overline{W}_S^{g(q)}$ and their underlying group schemes were investigated in some detail for $S = \mathbb{N}$. It is possible to prove directly that $W_S^{(q)}$ and \overline{W}_S^{1-q} together with their extra structures are isomorphic. However this also follows without effort from the universality property of $W_S^{(q)}$, c.f. Corollary 8 and the subsequent example.

3. A VARIANT OF THE CARTIER–DIEUDONNÉ LEMMA

In this section we prove two technical results which are the basis for the deformation theory of the Witt vector scheme in the next section. For a divisor stable subset T of \mathbb{N} we write $S \preccurlyeq T$ to signify that S is a divisor stable subset of T . Let $\underline{A} = (A_S)_{S \preccurlyeq T}$ be a projective system of rings on T i.e. a contravariant functor from the ordered set $\{S \preccurlyeq T\}$ viewed as a category to the category of (commutative) unital or non-unital rings. For $S_1 \preccurlyeq S_2 \preccurlyeq T$ we denote the transition maps simply by $\pi : A_{S_2} \rightarrow A_{S_1}$. We say that \underline{A} is equipped with commuting Frobenius lifts if for all prime numbers $p \in S \preccurlyeq T$ there are ring homomorphisms

$$F_p : A_S \longrightarrow A_{S/p}$$

with the following properties:

- 1) For all $a \in A_S$ we have the congruence

$$F_p(a) \equiv \pi(a)^p \bmod pA_{S/p} .$$

- 2) The F_p are natural in the sense that for $p \in S_1 \preccurlyeq S_2 \preccurlyeq T$ the diagram

$$\begin{array}{ccc} A_{S_2} & \xrightarrow{F_p} & A_{S_2/p} \\ \pi \downarrow & & \downarrow \pi \\ A_{S_1} & \xrightarrow{F_p} & A_{S_1/p} \end{array}$$

commutes.

- 3) For prime numbers l with $pl \in S \preccurlyeq T$, the diagram

$$\begin{array}{ccc} A_S & \xrightarrow{F_p} & A_{S/p} \\ F_l \downarrow & & \downarrow F_l \\ A_{S/l} & \xrightarrow{F_p} & A_{S/pl} \end{array}$$

commutes.

For each $n \in S$ we define $F_n : A_S \rightarrow A_{S/n}$ as the composition $F_n = F_{p_1}^{\nu_1} \circ \dots \circ F_{p_r}^{\nu_r}$ where $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$ is the prime decomposition of n . By 2) this is well defined and the naturality and commutation properties 2) and 3) then hold without the assumption that p and l are prime numbers. Morphisms of projective systems of rings on T with commuting Frobenius lifts are defined in the obvious way and we obtain a category \mathcal{RF}_T both in the unital and in the non-unital cases. Note that for a ring A , Witt vector theory gives us an object $W(A) := (W_S(A))_{S \preceq T}$ of \mathcal{RF}_T . Generalizing a well known fact from the theory of Witt vector rings we show that W has the following universal property:

Proposition 1. *Assume that for A in \mathcal{RF}_T the ring $A = A_{\{1\}}$ has no T -torsion. Then there is a unique morphism $\alpha = (\alpha_S)_{S \preceq T} : A \rightarrow W(A)$ in \mathcal{RF}_T with $\alpha_S = \text{id}_A$ for $S = \{1\}$. The morphism α is functorial in A . Explicitly it is given as follows. The composition of $\alpha_S : A_S \rightarrow W_S(A)$ with the (injective) ghost map $\log_S : W_S(A) \rightarrow A^S$ is given by the formula*

$$(\log_S \circ \alpha_S)(a) = (\pi F_n(a))_{n \in S} \quad \text{for } a \in A_S.$$

Here π denotes the map $\pi : A_{S/n} \rightarrow A_{\{1\}} = A$.

Proof. We have to show that there is a unique family of ring homomorphisms $\alpha_S : A_S \rightarrow W_S(A)$ for $S \preceq T$ which commute with the projections π and the Frobenius maps. Since A has no T -torsion by assumption, the ghost maps

$$\log_S : W_S(A) \longrightarrow A^S$$

give isomorphisms onto their images $X_S(A) := \log_S(W_S(A))$. On the ghost side the projection $\pi : X_{S_2}(A) \rightarrow X_{S_1}(A)$ for $S_1 \preceq S_2$ is induced by the projection $A^{S_2} \rightarrow A^{S_1}$. The Frobenius map

$$F_n : X_S(A) \longrightarrow X_{S/n}(A) \quad \text{for } n \in S \preceq T$$

is the restriction of the map

$$F_n : A^S \longrightarrow A^{S/n}, \quad (a_\nu)_{\nu \in S} \longmapsto (a_{nd})_{d \in S/n}.$$

The uniqueness assertion follows from the next claim:

Claim There is a unique family of ring homomorphisms

$$\beta_S : A_S \longrightarrow A^S \quad \text{for } S \preceq T, \text{ where } \beta_{\{1\}} = \text{id}$$

commuting with the respective projections π and Frobenius maps F_p . It is given by the formula

$$(3) \quad \beta_S(a) = (\pi F_n(a))_{n \in S} \quad \text{for } a \in A_S$$

where $\pi : A_{S/n} \rightarrow A_{\{1\}} = A$.

As for the uniqueness and formula (3) we argue as follows. Write $\beta_S(a) = (x_n)_{n \in S}$. For $n \in S$ set $S' = \{d \in S \mid d \mid n\}$. Then $S' \preccurlyeq S$ and $S'/n = \{1\}$. Consider the commutative diagram:

$$\begin{array}{ccccc} A_S & \xrightarrow{\pi} & A_{S'} & \xrightarrow{F_n} & A \\ \beta_S \downarrow & & \beta_{S'} \downarrow & & \downarrow \beta_{\{1\}} = \text{id} \\ A^S & \xrightarrow{\pi} & A^{S'} & \xrightarrow{F_n} & A. \end{array}$$

We find

$$\begin{aligned} x_n &= F_n((x_d)_{d \in S'}) = F_n \pi \beta_S(a) = F_n \pi(a) \\ &= \pi F_n(a). \end{aligned}$$

Here the final equality is due to the commutative diagram where $A_{S'/n} = A$

$$\begin{array}{ccc} A_S & \xrightarrow{\pi} & A_{S'} \\ F_n \downarrow & & \downarrow F_n \\ A_{S/n} & \xrightarrow{\pi} & A_{S'/n}. \end{array}$$

Thus $\beta = (\beta_S)$ must be given by (3). It is straightforward to check that (3) does indeed define the desired family of maps.

For the proof of the proposition it remains to show that $\beta_S(A_S) \subset X_S(A)$ for all $S \preccurlyeq T$. Thus for every $a \in A_S$ there have to be (uniquely determined) elements $a_d \in A$ for $d \in S$ with

$$(4) \quad \pi F_n(a) = \sum_{d \mid n} d a_d^{n/d} \quad \text{in } A \text{ for every } n \in S.$$

For this we prove the following stronger statement by induction on $n \in S$.

Claim Given $n \in S$ and $b \in A_S$, there exist elements $b_d \in A_{S/d}$ for all $d \mid n$ such that we have

$$(5) \quad F_n(b) = \sum_{d \mid n} d \pi(b_d)^{n/d} \quad \text{in } A_{S/n}.$$

Here $\pi(b_d)$ is the projection of b_d along $\pi : A_{S/d} \rightarrow A_{S/n}$.

For $n = 1$ we take $b_1 = b$. Assume that the claim holds for proper divisors of n . For any prime divisor p of n we then know that

$$F_{n/p}(b) = \sum_{d \mid (n/p)} d \pi(b_d)^{n/pd} \quad \text{in } A_{S/(n/p)}$$

for elements $b_d \in A_{S/d}$ and corresponding $\pi : A_{S/d} \rightarrow A_{S/(n/p)}$. Applying F_p we get

$$F_n(b) = \sum_{d \mid (n/p)} d \pi F_p(b_d)^{n/pd} \quad \text{in } A_{S/n}$$

where the π 's are projections $A_{S/dp} \rightarrow A_{S/n}$. By assumption

$$F_p(b_d) \equiv \pi(b_d)^p \bmod pA_{S/dp}$$

and therefore

$$F_p(b_d)^{n/pd} \equiv \pi(b_d)^{n/d} \bmod p^{v_p(n/d)} A_{S/dp}.$$

Here we have used the fact that $\alpha \equiv \beta \bmod p$ implies that $\alpha^{p^i} \equiv \beta^{p^i} \bmod p^{i+1}$ and hence $\alpha^k \equiv \beta^k \bmod p^{v_p(k)+1}$ for $k \geq 1$. It follows that

$$\begin{aligned} F_n(b) &\equiv \sum_{d|(n/p)} d\pi(b_d)^{n/d} \bmod p^{v_p(n)} A_{S/n} \\ &\equiv \sum_{\substack{d|n \\ d \neq n}} d\pi(b_d)^{n/d} \bmod p^{v_p(n)} A_{S/n}. \end{aligned}$$

Here and in the following the notation $d|n$ means that $d|n$ and $d \neq n$.

For the last step, note that if $d|n$ and $d \nmid (n/p)$ then $v_p(d) = v_p(n)$. Since $p|n$ was arbitrary we conclude that

$$F_n(b) \equiv \sum_{\substack{d|n \\ d \neq n}} d\pi(b_d)^{n/d} \bmod nA_{S/n}.$$

Hence an element $b_n \in A_{S/n}$ can be found so that (5) holds. The explicit formula for $\log_S \circ \alpha_S$ shows that $\alpha = (\alpha_S)$ depends functorially on \underline{A} . \square

Remark In the appendix we sketch a theory of Witt vector rings for ind-rings which elucidates the somewhat ad hoc proof of Proposition 1.

In general the morphism α in Proposition 1 will not be an isomorphism. For this more structure is required. We call a projective system $\underline{A} = (A_S)_{S \preceq T}$ *continuous* if for all $S \preceq T$ we have $A_S = \varprojlim_{S_0 \preceq S} A_{S_0}$ where S_0 runs over the *finite* divisor stable subsets of S . Note that for finite T continuity is automatic. Consider the category \mathcal{RFV}_T whose objects are continuous projective systems $\underline{A} = (A_S)_{S \preceq T}$ of rings on T with commuting Frobenius lifts F_p together with Verschiebung maps for all prime numbers $p \in S \preceq T$

$$V_p : A_{S/p} \longrightarrow A_S$$

with the following properties:

4) The V_p are additive homomorphisms which are natural in the sense that for $p \in S_1 \preceq S_2 \preceq T$ the diagram

$$\begin{array}{ccc} A_{S_2/p} & \xrightarrow{V_p} & A_{S_2} \\ \pi \downarrow & & \downarrow \pi \\ A_{S_1/p} & \xrightarrow{V_p} & A_{S_1} \end{array}$$

commutes.

5) For prime numbers l with $pl \in S \preccurlyeq T$ the diagram

$$\begin{array}{ccc} A_{S/pl} & \xrightarrow{V_p} & A_{S/l} \\ V_l \downarrow & & \downarrow V_l \\ A_{S/p} & \xrightarrow{V_p} & A_S \end{array}$$

commutes.

6) The composition $A_{S/p} \xrightarrow{V_p} A_S \xrightarrow{F_p} A_{S/p}$ is p -multiplication i.e. $F_p \circ V_p = p$ for $p \in S \preccurlyeq T$. For any prime l with $l \neq p$ and $pl \in S \preccurlyeq T$ the diagram

$$\begin{array}{ccc} A_{S/p} & \xrightarrow{V_p} & A_S \\ F_l \downarrow & & \downarrow F_l \\ A_{S/pl} & \xrightarrow{V_p} & A_{S/l} \end{array}$$

commutes.

7) There are exact sequences of additive groups

$$0 \longrightarrow A_{S/p} \xrightarrow{V_p} A_S \xrightarrow{\pi} A_{S(p)} \longrightarrow 0.$$

Here $S(p) = \{d \in S \mid p \nmid d\}$.

Morphisms between objects of \mathcal{RFV}_T are defined in the obvious way. For a ring A , Witt vector theory gives an object $W(A) = (W_S(A))_{S \preccurlyeq T}$ of \mathcal{RFV}_T .

Proposition 2. *Assume that for \underline{A} in \mathcal{RFV}_T the ring $A = A_{\{1\}}$ has no T -torsion. Then the morphism $\alpha : \underline{A} \rightarrow W(A)$ in \mathcal{RF}_T from Proposition 1 defines an isomorphism in \mathcal{RFV}_T . In particular it is an isomorphism in \mathcal{RF}_T .*

Proof. The main point is that for any prime $p \in S \preccurlyeq T$ the diagram

$$(6) \quad \begin{array}{ccc} A_{S/p} & \xrightarrow{V_p} & A_S \\ \alpha_{S/p} \downarrow & & \downarrow \alpha_S \\ W_{S/p}(A) & \xrightarrow{V_p} & W_S(A) \end{array}$$

commutes. Since A has no T -torsion, the ghost map \log_S on $W_S(A)$ is injective and hence it suffices to show that we have

$$(7) \quad \log_S \circ V_p \circ \alpha_{S/p} = \log_S \circ \alpha_S \circ V_p \quad \text{on } A_{S/p}.$$

For $a \in A_{S/p}$, using the explicit description of $\log_S \circ \alpha$ in Proposition 1 we find, setting $\delta_{p|n} = 1$ for $p \mid n$ and $= 0$ for $p \nmid n$:

$$\begin{aligned} (\log_S \circ \alpha_S \circ V_p)(a) &= (\log_S \circ \alpha_S)(V_p(a)) = (\pi F_n V_p(a))_{n \in S} \\ &= (\delta_{p|n} \pi F_n V_p(a))_{n \in S}. \end{aligned}$$

Namely, for $p \nmid n$ we have, by 6) and 7) above

$$\pi F_n V_p = \pi V_p F_n = 0.$$

Using 6) again we therefore find

$$\begin{aligned} (\log_S \circ \alpha_S \circ V_p)(a) &= (\delta_{p|n} \pi F_{n/p} F_p V_p(a))_{n \in S} \\ &= p(\delta_{p|n} \pi F_{n/p}(a))_{n \in S} \\ &= V_p((\pi F_n(a))_{n \in S/p}) \\ &= (V_p \circ \log_{S/p} \circ \alpha_{S/p})(a) \\ &= (\log_S \circ V_p \circ \alpha_{S/p})(a). \end{aligned}$$

Thus we have shown (7) and hence (6). Combining (6) and 7) we get a commutative diagram with exact lines for all $p \in S \preccurlyeq T$

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{S/p} & \xrightarrow{V_p} & A_S & \xrightarrow{\pi} & A_{S(p)} \longrightarrow 0 \\ & & \alpha_{S/p} \downarrow & & \alpha_S \downarrow & & \alpha_{S(p)} \downarrow \\ 0 & \longrightarrow & W_{S/p}(A) & \xrightarrow{V_p} & W_S(A) & \xrightarrow{\pi} & W_{S(p)}(A) \longrightarrow 0 \end{array}$$

Since $\alpha_{\{1\}} = \text{id}$, an induction with respect to $|S|$ now shows that α_S is an isomorphism for all finite S . The general case follows by the continuity of A and W . \square

4. UNIVERSALITY OF WITT VECTOR SCHEMES

As before, fix a divisor stable subset $T \subset \mathbb{N}$. Consider a projective system $\underline{M} = (M_S)_{S \preccurlyeq T}$ of affine commutative ring-schemes $M_S = \text{spec } B_S$ over a base ring R . As before, the transition maps are denoted by π . We say that \underline{M} is equipped with commuting Frobenius lifts if for all prime numbers $p \in S \preccurlyeq T$ there are morphisms of ring-schemes over R

$$F_p : M_S \longrightarrow M_{S/p}$$

such that for all R -algebras C , the projective system $\underline{M}(C) = (M_S(C))_{S \preccurlyeq T}$ together with the maps $F_p(C)$ is an object of \mathcal{RF}_T . In particular the relations $\pi \circ F_p = F_p \circ \pi$ and $F_p \circ F_l = F_l \circ F_p$ hold in the obvious sense, c.f. 2), 3) in section 3. The ensuing category is denoted by $\mathcal{RF}_{T/R}$. We may view its objects as the “representable functors” from the category of R -algebras to \mathcal{RF}_T . We define another category $\mathcal{RFV}_{T/R}$ as follows. Objects are continuous projective systems \underline{M} in $\mathcal{RF}_{T/R}$ equipped with morphisms of the underlying additive group schemes $V_p : M_{S/p} \rightarrow M_S$ for all prime numbers $p \in S \preccurlyeq T$ such that on R -algebras C one gets objects of \mathcal{RFV}_T . Morphisms are the evident ones. In particular $\pi \circ V_p = V_p \circ \pi$ and $V_p \circ V_l = V_l \circ V_p$ and $F_p \circ V_p = p \text{id}$, moreover $F_l \circ V_p = V_p \circ F_l$ for $p \neq l$ and the sequence

$$(8) \quad 0 \longrightarrow M_{S/p} \xrightarrow{V_p} M_S \xrightarrow{\pi} M_{S(p)} \longrightarrow 0$$

is exact and π is (scheme-theoretically) split. We may view the objects of $\mathcal{RFV}_{T/R}$ as the representable functors from the category of R -algebras to \mathcal{RFV}_T .

Given any affine ring-scheme $M = \text{spec } B$ over R , the functor

$$W_S^M = W_S \circ M : \{R\text{-algebras}\} \longrightarrow \{\text{rings}\}$$

is an affine ring scheme over R with underlying scheme $M^S = \text{spec } B^{\otimes S}$. The Witt-vector Frobenius and Verschiebung maps for all prime numbers $p \in S$

$$F_p : W_S(M(C)) \longrightarrow W_{S/p}(M(C)) \quad \text{and} \quad V_p : W_{S/p}(M(C)) \longrightarrow W_S(M(C))$$

for the rings $M(C)$ where C runs over R -algebras come by the Yoneda lemma from morphisms of ring schemes

$$F_p : W_S^M \longrightarrow W_{S/p}^M \quad \text{and} \quad V_p : W_{S/p}^M \longrightarrow W_S^M .$$

By the usual Witt-vector theory we see that equipped with the F_p 's (and V_p 's) the projective system of ring-schemes $\underline{W}^M := (W_S^M)_{S \preceq T}$ becomes an object of $\mathcal{RF}_{T/R}$ resp. $\mathcal{RFV}_{T/R}$.

We need a technical condition in the following:

Definition An object \underline{M} in $\mathcal{RF}_{T/R}$ or $\mathcal{RFV}_{T/R}$ has no Hopf T -torsion if for $M = M_{\{1\}}$ the abelian groups $M(R)$, $M(B_S)$ and $M(B_S \otimes_{\mathbb{Z}} B_S)$ have no T -torsion for all $S \preceq T$ where $M_S = \text{spec } B_S$.

Theorem 3. a) Assume that \underline{M} in $\mathcal{RF}_{T/R}$ has no Hopf T -torsion. Then there is a unique morphism

$$\alpha = (\alpha_S)_{S \preceq T} : \underline{M} \rightarrow \underline{W}^M$$

in $\mathcal{RF}_{T/R}$ with $\alpha_S = \text{id}$ for $S = \{1\}$.

b) Assume that \underline{M} in $\mathcal{RFV}_{T/R}$ has no Hopf T -torsion. Then the above unique morphism $\alpha : \underline{M} \rightarrow \underline{W}^M$ in $\mathcal{RF}_{T/R}$ defines an isomorphism in $\mathcal{RFV}_{T/R}$ (and hence in $\mathcal{RF}_{T/R}$).

Proof. For the class \mathcal{C} of R -algebras C with $M(C)$ T -torsion free, Propositions 1 and 2 ensure the existence of unique morphisms in \mathcal{RF}_T (resp. isomorphisms in \mathcal{RFV}_T)

$$\alpha(C) = (\alpha(C)_S)_{S \preceq T} : M(C) \longrightarrow \underline{W}^M(C) = \underline{W}(M(C)) .$$

Moreover they are functorial in $C \in \mathcal{C}$. We only need to show that the functorial ring homomorphisms $\alpha(C)_S : M_S(C) \rightarrow W_S^M(C)$ come from uniquely determined morphisms of ring-schemes $\alpha_S : M_S \rightarrow W_S^M$ which commute with the maps π, F_p, V_p 's. Since \underline{M} has no Hopf T -torsion, the existence of α_S follows from the next version of the Yoneda lemma applied to $F = M_S, G = W_S^M$ and the class \mathcal{C} above. Another application of (the first part of) the lemma shows that the α_S 's commute with π, F_p, V_p . \square

Lemma 4. *Let $F = \operatorname{spec} A$ and $G = \operatorname{spec} B$ be two affine ring-schemes over R . Assume that for a class \mathcal{C} of R -algebras there are functorial ring homomorphisms for all $C \in \mathcal{C}$,*

$$\alpha(C) : F(C) \longrightarrow G(C) .$$

If $A \in \mathcal{C}$, then there is a unique morphism of R -schemes $\alpha : F \rightarrow G$ which induces $\alpha(C)$ for all $C \in \mathcal{C}$. If in addition $R \in \mathcal{C}$ and $A \otimes_R A \in \mathcal{C}$, then α is a morphism of ring-schemes.

Proof. The morphism $\operatorname{spec} \alpha^\sharp : F \rightarrow G$ induced by

$$\alpha^\sharp = \alpha(A)(\operatorname{id}_A) \in \operatorname{Hom}_{R\text{-alg}}(B, A)$$

induces the given maps $\alpha(C)$, since for $\psi \in F(C) = \operatorname{Hom}_{R\text{-alg}}(A, C)$ we have

$$\alpha(C)(\psi) = \psi \circ \alpha^\sharp = (\operatorname{spec} \alpha^\sharp)(\psi) .$$

On the other hand, this equation applied to $C = A, \psi = \operatorname{id}$ implies that $\alpha^\sharp = \alpha(A)(\operatorname{id})$ is unique, hence $\alpha := \operatorname{spec} \alpha^\sharp$ is unique as well. Since $\alpha(C)$ is an additive (multiplicative) map for $C \in \mathcal{C}$ and since $\alpha(C) = (\alpha^\sharp)^*, \alpha(C) \times \alpha(C) = (\alpha^\sharp \otimes \alpha^\sharp)^*$, we have

$$(\alpha^\sharp)^* \circ \Delta^* = \Delta^* \circ (\alpha^\sharp \otimes \alpha^\sharp)^*$$

on $F(C) \times F(C) = \operatorname{Hom}_{R\text{-alg}}(A \otimes A, C)$ where Δ is the co-addition (co-multiplication) on A resp. B . If $A \otimes A \in \mathcal{C}$, we can apply this to $C = A \otimes A$ and $\operatorname{id} \in \operatorname{Hom}(A \otimes A, A \otimes A)$ and obtain

$$\Delta \circ \alpha^\sharp = (\alpha^\sharp \otimes \alpha^\sharp) \circ \Delta .$$

If $R \in \mathcal{C}$, then $\alpha(R) = (\alpha^\sharp)^*$ and from $\alpha(R)(0) = 0$ we get $(\alpha^\sharp)^*(\varepsilon_A) = \varepsilon_B$ i.e. $\varepsilon_B = \varepsilon_A \circ \alpha^\sharp$ for the co-zeroes (or co-units) ε_A and ε_B of A and B . \square

We end this section with an application of Theorem 3. Two divisor stable subsets T_1 and T_2 of \mathbb{N} are coprime if and only if $T_1 \cap T_2 = \{1\}$. In this case the set $T_1 \cdot T_2 = \{nm \mid n \in T_1, m \in T_2\}$ is again divisor stable. Lenstra conjectured and in [A] Auer proved that there are natural isomorphisms of functors on unital rings

$$(9) \quad W_{T_1 \cdot T_2} \xrightarrow{\sim} W_{T_1} \circ W_{T_2} .$$

Let us explain how this follows from Theorem 3. For $S \preccurlyeq T_1$ set

$$M_S = W_{S \cdot T_2} .$$

For $n \in S$ we have $(S \cdot T_2)/n = (S/n) \cdot T_2$. Hence the usual Witt vector Frobenius and Verschiebung morphisms give morphisms

$$F_n : M_S \longrightarrow M_{S/n} \quad \text{and} \quad V_n : M_{S/n} \longrightarrow M_S .$$

Using them we obtain an object $\tilde{M} = (M_S)_{S \preccurlyeq T_1}$ of $\mathcal{RFV}_{T_1/\mathbb{Z}}$ with $M := M_{\{1\}} = W_{T_2}$. The coordinate rings B_S of all M_S are polynomial algebras over \mathbb{Z} and so are $B_S \otimes_{\mathbb{Z}} B_S$. The ring $W_{T_2}(A)$ is a subring of A^{T_2} via the ghost

map if A has no T_2 -torsion. Therefore if A has no \mathbb{Z} -torsion it follows that M has no Hopf T_1 -torsion. Hence we can apply Theorem 3 b) and obtain a uniquely determined isomorphism

$$\alpha : M = (W_{S \cdot T_2})_{S \preccurlyeq T_1} \longrightarrow W^M = (W_S \circ W_{T_2})_{S \preccurlyeq T_1}$$

in $\mathcal{RFV}_{T/\mathbb{Z}}$ with $\alpha_S = \text{id}$ for $S = \{1\}$. In particular we get a natural isomorphism

$$(10) \quad \alpha_{T_1} : W_{T_1 \cdot T_2} \xrightarrow{\sim} W_{T_1} \circ W_{T_2} .$$

The Witt vector functor W_S transforms short exact sequences

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

of rings into short exact sequences of rings

$$0 \longrightarrow W_S(I) \longrightarrow W_S(A) \longrightarrow W_S(B) \longrightarrow 0 .$$

Hence the isomorphism (10) is also valid on non-unital rings (they are kernels of surjections of unital rings). Applying (10) to $A^{(q)}$ for $\mathbb{Z}[q]$ -algebras A we obtain an isomorphism of ring-schemes over $\text{spec } \mathbb{Z}[q]$

$$W_{T_1 \cdot T_2}^{(q)} \xrightarrow{\sim} W_{T_1} \circ W_{T_2}^{(q)} .$$

Note that for $T_2 = \{1\}$ this holds by definition.

5. DEFORMATIONS OF WITT VECTOR SCHEMES

In this section we prove universality over reduced bases for the q -deformation $W^{(q)}$ introduced in section 2 and show that $W^{(q)}$ is isomorphic to \overline{W}^{1-q} .

For an R -algebra C and an element $r \in R$ we denote by $C^{(r)}$ the ring with underlying additive group $(C, +)$ and twisted multiplication $x * y = rxy$. The ring $C^{(r)}$ is unital if and only if $r \in R^\times$. In this case $1_{C^{(r)}} = r^{-1}$ is the unity. Let $\mathbb{G}_R^{(r)}$ be the ring-scheme over R defined by $\mathbb{G}_R^{(r)}(C) = C^{(r)}$ for all R -algebras C . It is represented by the polynomial algebra $R[t]$ together with

$$\Delta_+(t) = t \otimes 1 + 1 \otimes t \quad \text{and} \quad \Delta_\bullet(t) = rt \otimes t .$$

Moreover $\varepsilon_0(t) = 0$ and $\mathbb{G}_R^{(r)}$ is unital if and only if $r \in R^\times$. In this case the co-unit ε_1 is given by $\varepsilon_1(t) = r^{-1}$.

In particular we have a non-unital ring-scheme $\mathbb{G}^{(q)}$ over $\mathbb{Z}[q]$ and a unital ring-scheme $\mathbb{G}^{(q)}$ over $\mathbb{Z}[q, q^{-1}]$. An element $r \in R$ corresponds to a ring homomorphism $\mathbb{Z}[q] \rightarrow R$ and we have

$$(11) \quad \mathbb{G}_R^{(r)} = \mathbb{G}^{(q)} \otimes_{\mathbb{Z}[q]} R$$

as non-unital ring-schemes over R . Similarly a unit $r \in R^\times$ determines a ring homomorphism $\mathbb{Z}[q, q^{-1}] \rightarrow R$ and

$$\mathbb{G}_R^{(r)} = \mathbb{G}^{(q)} \otimes_{\mathbb{Z}[q, q^{-1}]} R$$

as unital ring-schemes. For any two units $r, r' \in R^\times$ the ring-schemes $\mathbb{G}_R^{(r)}$ and $\mathbb{G}_R^{(r')}$ are isomorphic in the category of unital ring-schemes. Now let $r, r' \in R$ be arbitrary. In the category of non-unital ring-schemes $\mathbb{G}_R^{(r)}$ and $\mathbb{G}_R^{(r')}$ are isomorphic if and only if $r' = ru$ for some $u \in R^\times$. There is a natural bijection

$$(12) \quad \{u \in R^\times \mid r = r'u\} \xrightarrow{\sim} \text{Iso}(\mathbb{G}_R^{(r)}, \mathbb{G}_R^{(r')})$$

and in particular an isomorphism of groups

$$\{u \in R^\times \mid r = ru\} \xrightarrow{\sim} \text{Aut}(\mathbb{G}_R^{(r)}) .$$

Here Iso and Aut are taken in the category of non-unital ring-schemes over R . If $r \in R$ is not a zero-divisor then $\text{Aut}(\mathbb{G}_R^{(r)}) = \{\text{id}\}$.

We denote by $(\mathbb{G}_R^{(r)}, t)$ the pair consisting of $\mathbb{G}_R^{(r)}$ and the chosen local coordinate t above. Consider two ring-schemes M_1 and M_2 over R whose underlying schemes are isomorphic to \mathbb{A}^1 and which are enhanced by coordinates z_1 and z_2 in zero. We say that (M_1, z_1) and (M_2, z_2) are isomorphic if there is an isomorphism (necessarily unique) of ring-schemes $\varphi : M_1 \rightarrow M_2$ with $z_1 = z_2 \circ \varphi$. Clearly $(\mathbb{G}_R^{(r)}, t) \cong (\mathbb{G}_R^{(r')}, t)$ if and only if $r = r'$.

Proposition 5. *Let R be a reduced ring and let M be a ring-scheme over R with underlying scheme isomorphic to \mathbb{A}^1 and coordinate z in zero. Then there exists a unique element $r \in R$ such that $(M, z) \cong (\mathbb{G}_R^{(r)}, t)$. If M is unital and we work in the category of unital ring-schemes, then $r \in R^\times$.*

Proof. We have $M = \text{spec } R[z]$. The ring-scheme structure of M corresponds to co-addition Δ_+ , co-multiplication Δ_\bullet , co-zero ε_0 and in the unital case a co-unit ε_1 . By the choice of z we have $\varepsilon_0(z) = 0$. Set $F(X, Y) = \Delta_+(z)$ and $G(X, Y) = \Delta_\bullet(z)$ where we set $X = z \otimes 1$ and $Y = 1 \otimes z$. Then F, G determine a one-dimensional polynomial ring law over R . Now the assertion follows from the next lemma. \square

Lemma 6. *Let R be a reduced ring and $F, G \in R[X, Y]$ a one-dimensional polynomial ring law over R . Then there is a unique element $r \in R$ such that*

$$F(X, Y) = X + Y \quad \text{and} \quad G(X, Y) = rXY .$$

Proof. Let $n \geq 1$ be the highest power of x in $F(x, y)$. An inspection of the formula

$$F(x, F(y, z)) = F(F(x, y), z)$$

shows that $n = n^2$ since R has no nilpotent elements except 0. Hence $n = 1$ and similarly for y . Thus

$$F(x, y) = x + y + cxy \quad \text{for some } c \in R .$$

By assumption there is a polynomial $I(x) = -x + \deg \geq 2$ such that $F(x, I(x)) = 0$. Since R is reduced, it follows that $c = 0$. Hence $F(x, y) = x + y$. Associativity of G implies similarly as for F that $G(x, y) = rxy$ for some $r \in R$. \square

Remark Over non-reduced bases there are more one-dimensional polynomial ring laws than the ones in the lemma. Over $R = \mathbb{Z}[\varepsilon]/(\varepsilon^2)$ for example the polynomials $F(X, Y) = X + Y + \varepsilon XY$ and $G(X, Y) = r\varepsilon XY$ for $r \in R$ define a different class.

Since we are interested in deformations of Witt vector schemes it is natural to introduce the full subcategory $\mathcal{AFV}_{T/R}$ of $\mathcal{RFV}_{T/R}$ whose objects \underline{M} satisfy $M_{\{1\}} \cong \mathbb{A}^1$ as schemes. We rigidify $\mathcal{AFV}_{T/R}$ by considering the category $\mathcal{AFV}_{T/R}^+$ of pairs (\underline{M}, z) where z is a coordinate in zero of $M_{\{1\}}$. A morphism $\varphi : (\underline{M}, z) \rightarrow (\underline{M}', z')$ is a morphism $\varphi : \underline{M} \rightarrow \underline{M}'$ in $\mathcal{AFV}_{T/R}^+$ such that $z = z' \circ \varphi_{\{1\}}$. In particular $\varphi_{\{1\}} : M_{\{1\}} \rightarrow M'_{\{1\}}$ is an isomorphism.

For $S \preccurlyeq T$ recall the non-unital ring scheme $W_S^{(q)}$ over $\mathbb{Z}[q]$ from section 2. We have $W_S^{(q)} = W_S^{\mathbb{G}^{(q)}}$ and $W_{\{1\}}^{(q)} = \mathbb{A}^1$ is enhanced by the fixed coordinate t . Thus we may view $W^{(q)} := W^{\mathbb{G}^{(q)}}$ as objects of $\mathcal{AFV}_{T/\mathbb{Z}[q]}$ and $\mathcal{AFV}_{T/\mathbb{Z}[q]}^+$. Similar definitions over the ring $\mathbb{Z}[q, q^{-1}]$ hold in the unital case.

Corollary 7. *Let R be a reduced ring without T -torsion and \underline{M} an object of $\mathcal{AFV}_{T/R}^+$. In the non-unital case there is a unique homomorphism $\mathbb{Z}[q] \rightarrow R$ such that there exists a morphism in $\mathcal{AFV}_{T/R}^+$*

$$\alpha : \underline{M} \longrightarrow W^{(q)} \otimes_{\mathbb{Z}[q]} R.$$

The morphism α is uniquely determined and it is an isomorphism. In the unital case the corresponding assertions hold with $\mathbb{Z}[q]$ replaced by $\mathbb{Z}[q, q^{-1}]$.

Proof. Since the sequence (8) is scheme-theoretically split and since $M = M_{\{1\}}$ is isomorphic to \mathbb{A}^1 , it follows inductively that $M_S \cong \mathbb{A}^S$ as schemes if S is finite. Continuity of \underline{M} implies that $M_S \cong \mathbb{A}^S$ for all $S \preccurlyeq T$. Writing $M_S = \text{spec } B_S$, the algebra B_S is therefore a polynomial algebra over R . Since R has no T -torsion by assumption and since $M \cong \mathbb{A}^1$ we see that \underline{M} has no Hopf T -torsion. Now the corollary follows from Theorem 3 and Proposition 5. \square

Remark For fixed T it follows that $(W^{(q)}, t)$ over $\text{spec } \mathbb{Z}[q]$ (resp. $\text{spec } \mathbb{Z}[q, q^{-1}]$) is the universal deformation over reduced bases of (W, t) in \mathcal{AFV}_T^+ .

We say that a ring homomorphism $\varphi : \mathbb{Z}[q] \rightarrow R$ is determined up to a unit in R if $\varphi(q)$ is determined up to a unit in R . Forgetting the coordinate of M we get

Corollary 8. *Let R be a reduced ring without T -torsion and \underline{M} an object of the category $\mathcal{AFV}_{T/R}$.*

a) In the non-unital case there is a homomorphism $\varphi : \mathbb{Z}[q] \rightarrow R$ which is determined up to a unit in R such that there is an isomorphism in $\mathcal{RFV}_{T/R}$

$$\alpha : \underline{M} \rightarrow \underline{W}^{(q)} \otimes_{\mathbb{Z}[q], \varphi} R .$$

If $\varphi(q)$ is not a zero-divisor in R then α is uniquely determined. In the general case the set of α 's is parametrized by the stabilizer of $\varphi(q)$ under the action of R^\times on R .

b) In the unital case there is a unique isomorphism in $\mathcal{AFV}_{T/R}$

$$\alpha : \underline{M} \longrightarrow \underline{W} \otimes_{\mathbb{Z}} R .$$

Thus \underline{W} has no non-trivial deformations over reduced bases in \mathcal{AFV}_T .

Example 9. Recall the second family $\overline{W}_S^{g(q)}$ of q -deformed Witt vector schemes over $\text{spec } \mathbb{Z}[q]$ from section 2. Together with their Frobenius and Verschiebung structures and the first coordinate they make

$$\underline{\overline{W}}^{g(q)} := (\overline{W}_S^{g(q)})_{S \preccurlyeq T}$$

an object of $\mathcal{AFV}_{T/\mathbb{Z}[q]}^+$. We have $\overline{W}_{\{1\}}^{g(q)} = \mathbb{G}^{(1-g(q))}$ by construction and this identification respects the preferred coordinates at zero of both sides.

It follows from Corollary 7 that there is a unique isomorphism

$$\underline{\overline{W}}^{g(q)} \xrightarrow{\sim} \underline{W}^{(1-g(q))} \quad \text{in } \mathcal{AFV}_{T/\mathbb{Z}[q]}^+ .$$

Moreover this is the only isomorphism of $\underline{\overline{W}}^{g(q)}$ with $\underline{W}^{(r)}$ for some $r \in \mathbb{Z}[q]$ in this category. Forgetting the coordinate at zero and working in the category $\mathcal{AFV}_{T/\mathbb{Z}[q]}$, Corollary 8 implies that there are exactly two values of $r \in \mathbb{Z}[q]$ for which an isomorphism (automatically unique) $\overline{W}^{g(q)} \xrightarrow{\sim} W^{(r)}$ exists, namely $r = 1 - g(q)$ and $r = g(q) - 1$. Note here that $\mathbb{Z}[q]$ is an integral domain with $\mathbb{Z}[q]^\times = \{\pm 1\}$. In particular it follows that $\overline{W}_S^{g(q)} \cong \overline{W}_S^{2-g(q)}$ as observed in [O2] Proposition 3.9.

Example 10. In [L] Lenart introduced modified Witt rings $W^q(A)$ for every integer $q \in \mathbb{Z}$. In [O1] truncated versions $W_S^q(A)$ of these rings were studied. They are obtained as follows. Define $W_S^q(A) = A^S$ as sets and consider the ghost map

$$\Phi_S^q : W_S^q(A) \longrightarrow A^S ,$$

given by the formula

$$\Phi_S^q((a_d)_{d \in S}) = \left(\sum_{d|n} dq^{\frac{n}{d}-1} a_d^{\frac{n}{d}} \right)_{n \in S} .$$

It is the same ghost map as for $W_S^{(q)}(A)$ in formula (2) but on the ghost side A^S is taken componentwise with the usual ring-structure (instead of $(A^{(q)})^S$)

as in (2)). Using Fermat's little theorem it is shown in [L], [O1] that for any $q \in \mathbb{Z}$ there is a unique, possibly non-unital ring-structure on $W_S^q(A)$ which is functorial in A such that Φ_S^q becomes a ring homomorphism. Moreover the usual Frobenius and Verschiebung operators for $n \in S$ on the ghost side

$$F_n((a_\nu)_{\nu \in S}) = (a_{n\nu})_{\nu \in S/n} \quad \text{and} \quad V_n((a_\nu)_{\nu \in S/n}) = (n\delta_{n|\nu}a_{\nu/n})_{\nu \in S},$$

induce corresponding morphisms on the Lenart–Witt side. It is immediate that all conditions for $W^q := (W_S^q)_{S \preccurlyeq T}$ to be an object of $\mathcal{AFV}_{T/\mathbb{Z}}^+$ are satisfied except possibly for the property that F_p should reduce mod p to the p -th power map for all primes $p \in S \preccurlyeq T$. It can be shown with some effort that this condition is satisfied if and only if no prime divisor of q is contained in T . Since $W_{\{1\}}^q(A) = A$ as rings for any $q \in \mathbb{Z}$, it follows from Corollary 7 that W^q is uniquely isomorphic to W in $\mathcal{AFV}_{T/\mathbb{Z}}^+$ if no prime divisor of q is in T . Thus in this case there are natural isomorphisms of rings

$$W_S^q(A) \xrightarrow{\sim} W_S(A).$$

This fact is a special case of [O1] Theorem 14. If T contains a prime divisor of q then our theory does not apply to W^q . Let us illustrate the preceding discussion with the case $T = \{1, p\}$ where the calculations are easy.

The ghost maps are $\Phi_{\{1\}}^q = \text{id}$ and

$$\Phi_{\{1,p\}}^q(a_1, a_p) = (a_1, pa_p + q^{p-1}a_1^p).$$

Hence $W_{\{1\}}^q(A) = A$ as rings and on $W_{\{1,p\}}^q(A)$ addition and multiplication are given as follows

$$(a_1, a_p) + (b_1, b_p) = \left(a_1 + b_1, a_p + b_p - q^{p-1} \sum_{\nu=1}^{p-1} \frac{1}{p} \binom{p}{\nu} a_1^\nu b_1^{p-\nu} \right)$$

and

$$(a_1, a_p) \cdot (b_1, b_p) = (a_1 b_1, pa_p b_p + q^{p-1}(a_p b_1^p + a_1^p b_p) + \frac{1}{p} q^{p-1}(q^{p-1} - 1)a_1^p b_1^p).$$

The Frobenius morphism

$$F_p : W_{\{1,p\}}^q(A) \longrightarrow W_{\{1\}}^q(A) = A$$

is given by the formula

$$F_p(a_1, a_p) = pa_p + q^{p-1}a_1^p.$$

With the projection

$$\pi : W_{\{1,p\}}^q(A) \longrightarrow W_{\{1\}}^q(A) = A, \quad a = (a_1, a_p) \longmapsto a_1$$

we therefore have

$$F_p(a) \equiv q^{p-1}\pi(a)^p \pmod{pA}.$$

Thus the relation

$$F_p(a) \equiv \pi(a)^p \pmod{pA}$$

for all rings A is equivalent to $q^{p-1} \equiv 1 \pmod{p}$ i.e. to the assertion that $p \in T = \{1, p\}$ is not a prime divisor of q . In this case the map

$$\alpha : W_{\{1,p\}}^q(A) \xrightarrow{\sim} W_{\{1,p\}}(A)$$

with

$$\alpha(a_1, a_p) = \left(a_1, a_p + \frac{q^{p-1} - 1}{p} a_1^p \right)$$

is an isomorphism of rings.

APPENDIX: WITT VECTORS OF INDUCTIVE SYSTEMS OF RINGS

We sketch a natural generalization of the theory of Witt vectors to ind-rings.

Let $S \subset \mathbb{N}$ be a divisor stable subset and $\underline{A} = (A_n)_{n \in S}$ an inductive system of unital or non-unital commutative rings. This means that for $n \in S$ and $d \mid n$ there are ring homomorphisms

$$\pi_{d,n} : A_d \rightarrow A_n$$

with $\pi_{n,n} = \text{id}$ and $\pi_{d_1,n} = \pi_{d,n} \circ \pi_{d_1,d}$ if $d_1 \mid d$.

Consider the set

$$W_S(\underline{A}) = \prod_{n \in S} A_n$$

and the ghost map

$$\mathcal{G}_S : W_S(\underline{A}) \longrightarrow \prod_{n \in S} A_n$$

defined by

$$\mathcal{G}_{S,n}((a_\nu)_{\nu \in S}) = \sum_{d \mid n} d \pi_{d,n}(a_d)^{n/d} \quad \text{in } A_n .$$

Proposition 11. *Assume that A_n has no n -torsion for every $n \in S$. Then $\mathcal{G}_S : W_S(\underline{A}) \rightarrow \prod_{n \in S} A_n$ is injective.*

Proof. Assume that

$$\mathcal{G}_S((a_\nu)) = \mathcal{G}_S((b_\nu)) .$$

By definition we get $a_1 = b_1$. For $n \in S, n \neq 1$ assume that $a_d = b_d$ in A_d has been shown for all $d \mid n$. The equation $\mathcal{G}_{S,n}((a_\nu)) = \mathcal{G}_{S,n}((b_\nu))$ gives

$$n(a_n - b_n) = \sum_{\substack{d \mid n \\ d \neq n}} d(\pi_{d,n}(b_d)^{n/d} - \pi_{d,n}(a_d)^{n/d}) = 0$$

and hence $a_n = b_n$ since A_n has no n -torsion. □

Example If $(x_n) = \mathcal{G}_S((a_\nu))$ and $p \in S$, then

$$x_p = \pi_{1,p}(a_1)^p + p a_p \quad \text{and hence } x_p \equiv \pi_{1,p}(a_1)^p \pmod{p A_p} .$$

The Witt polynomials Σ_n and Π_n for addition and multiplication depend only on the variables x_d with $d \mid n$. Hence we can define addition and multiplication on $W_S(\underline{A})$ by setting

$$\underline{a} \oplus \underline{b} = \underline{c} \quad \text{for } \underline{a}, \underline{b} \in W_S(\underline{A})$$

where

$$c_n = \Sigma_n(\pi_{d,n}(a_d), \pi_{d,n}(b_d); d \mid n)$$

and similarly for multiplication. As in the usual case this is the only ring structure on the set $W_S(\underline{A})$ which is functorial in \underline{A} and for which \mathcal{G}_S is a ring homomorphism if $\prod_{n \in S} A_n$ is equipped with componentwise addition and multiplication. Similarly the polynomials defining the usual Frobenius morphisms $F_n : W_S \rightarrow W_{S/n}$ define commuting ring homomorphisms

$F_n : W_S(\underline{A}) \rightarrow W_{S/n}(F_n(\underline{A}))$ where $F_n(\underline{A}) = (A_{n\nu})_{\nu \in S/n}$. There are also additive Verschiebung maps

$$V_n : W_{S/n}(V_n(\underline{A})) \rightarrow W_S(\underline{A})$$

where $V_n(\underline{A}) = (A_\nu)_{\nu \in S/n}$ and $V_n((a_\nu)_{\nu \in S/n}) = (\delta_{n|\mu} \pi_{\mu/n, \mu}(a_{\mu/n}))_{\mu \in S}$ with $\delta_{n|\mu} = 1$ if $n \mid \mu$ and $\delta_{n|\mu} = 0$ if $n \nmid \mu$. The projection

$$\text{res} : W_S(\underline{A}) = \prod_{n \in S} A_n \xrightarrow{\text{pr}_1} A_1$$

is a surjective homomorphism of rings.

Example A ring A can be viewed as the inductive system A_{const} where $A_n = A$ and $\pi_{d,n} = \text{id}$ for $d \mid n, n \in S$. Then $W_S(A_{\text{const}}) = W_S(A)$ together with Frobenius and Verschiebung maps. We can also form the trivial inductive system A_{triv} of non-unital rings with $A_n = A$ and $\pi_{n,n} = \text{id}, \pi_{d,n} = 0$ for $d \nmid n$. Let $A^{(n)} = A$ as an additive group but with ring structure $a * b = nab$. Then $W_S(A_{\text{const}}) = \prod_{\nu \in S} A^{(\nu)}$, the product ring. The Frobenius map F_n corresponds to the map

$$F_n : \prod_{\nu \in S} A^{(\nu)} = W_S(A_{\text{const}}) \rightarrow W_S(F_n A_{\text{const}}) = \prod_{\nu \in S/n} A^{(\nu)}$$

with

$$F_n((a_\nu)_{\nu \in S}) = (na_{\nu n})_{\nu \in S/n}.$$

The ring $W(A_{\text{const}})$ appeared previously as $W^0(A)$ in [L] Corollary 5.9 (1).

For a divisor stable subset $T \subset S$ and an inductive system $\underline{A} = (A_n)_{n \in S}$ there is a natural surjective ring homomorphism

$$W_S(\underline{A}) \xrightarrow{\text{proj}} W_T(\text{res}_S^T(\underline{A})), (a_n)_{n \in S} \mapsto (a_n)_{n \in T}$$

where $\text{res}_S^T(\underline{A}) = (A_\nu)_{\nu \in T}$.

Proposition 12. *Consider an inductive system $\underline{A} = (A_n)_{n \in S}$. Then the following diagram is commutative:*

$$\begin{array}{ccccc} W_S(\underline{A}) & \xrightarrow{F_p} & W_{S/p}(F_p(\underline{A})) & \longrightarrow & W_{S/p}(F_p(\underline{A}))/p \\ \downarrow & & & & \uparrow W_{S/p}(\pi) \\ W_S(\underline{A})/p & \xrightarrow{(\cdot)^p} & W_S(\underline{A})/p & \longrightarrow & W_{S/p}(\text{res}_S^{S/p}(\underline{A}))/p \end{array}$$

Here $\pi : \text{res}_S^{S/p}(\underline{A}) \rightarrow F_p(\underline{A})$ is the map with ν -th component $\pi_{\nu, p\nu} : A_\nu \rightarrow A_{p\nu}$ for $\nu \in S/p$.

Proof. Set $\Lambda = \mathbb{Z}[x_d \mid d \in S]$ and let $(f_\nu)_{\nu \in S/p}$ with $f_\nu \in \mathbb{Z}[x_d \mid d|p\nu] \subset \Lambda$ be the family of polynomials defining the morphism $F_p : W_S \rightarrow W_{S/p}$. Also let $(G_\mu)_{\mu \in S}$ with $G_\mu \in \mathbb{Z}[x_d \mid d|\mu] \subset \Lambda$ be the family of polynomials defining the p -th power morphism $(\)^p : W_S \rightarrow W_S$. It is known that for $\nu \in S/p$ the difference $f_\nu - G_\nu$ is divisible by p in Λ . This is equivalent to the fact that diagram (1) commutes. Now for $(a_\mu)_{\mu \in S} \in W_S(\underline{A})$ the ν -th component for $\nu \in S/p$ of

$$F_p((a_\mu)_{\mu \in S}) - W_{S/p}(\pi)(\pi_{S, S/p}((a_\mu)_{\mu \in S}^p))$$

is given by

$$f_\nu(\pi_{d, p\nu}(a_d); d \mid p\nu) - G_\nu(\pi_{d, p\nu}(a_d); d \mid p\nu) .$$

This holds because the transition maps are ring homomorphisms and $\pi_{\nu, p\nu} \circ \pi_{d, \nu} = \pi_{d, p\nu}$. The assertion follows. \square

There is a Dwork lemma for our Witt vector rings of inductive systems.

Dwork lemma Let $\underline{A} = (A_n)_{n \in S}$ be an inductive system of rings on a divisor stable subset S of \mathbb{N} . Assume that for all $n \in S$ and primes $p \mid n$ we are given ring homomorphisms (compatible with the transition maps)

$$\phi_p : A_{n/p} \rightarrow A_n$$

such that the following diagram commutes:

$$\begin{array}{ccccc} A_{n/p} & \xrightarrow{\phi_p} & A_n & \longrightarrow & A_n/p \\ \downarrow & & & & \parallel \\ A_{n/p}/p & \xrightarrow{(\)^p} & A_{n/p}/p & \xrightarrow{\pi_{n/p, n}} & A_n/p . \end{array}$$

Thus, for $a \in A_{n/p}$ we have in A_n

$$\phi_p(a) \equiv \pi_{n/p, n}(a^p) \bmod pA_n .$$

Then the image of the ghost map

$$\mathcal{G}_S : W_S(\underline{A}) \rightarrow \prod_{n \in S} A_n$$

is the following subring:

$$\mathcal{G}_S(W_S(\underline{A})) = \{(x_n)_{n \in S} \mid \phi_p(x_{n/p}) \equiv x_n \bmod p^{v_p(n)} A_n \text{ for } p \mid n, n \in S\} .$$

Proof. The argument in the proof of Lemma 1.1 in [H] can be easily adapted to our setting. Let us look at the case $S = \{1, p\}$ for example. If $(x_n) = \mathcal{G}_S((a_\nu))$, then $x_1 = a_1$ and $x_p = \pi_{1, p}(x_1^p) + pa_p$. Hence $(x_1, x_p) \in \mathcal{G}_S(W_S(\underline{A}))$ if and only if $x_p \equiv \pi_{1, p}(x_1^p) \bmod pA_p$ or equivalently, if and only if $x_p \equiv \phi_p(x_1) \bmod pA_p$. \square

We need the following construction. Given an inductive system of rings $\underline{A} = (A_n)_{n \in S}$, we obtain another such system $\underline{W}(\underline{A})$ by setting

$$\underline{W}(\underline{A})_n = W_{S/n}(F_n(\underline{A})) \quad \text{for } n \in S$$

and defining $\pi_{d,n}$ to be the composition

$$\pi_{d,n} : W_{S/d}(F_d(\underline{A})) \xrightarrow{\text{proj}} W_{S/n}(\text{res}_{S/d}^{S/n}(F_d(\underline{A}))) \xrightarrow{W_{S/n}(\pi)} W_{S/n}(F_n(\underline{A})) .$$

Here

$$\pi : \text{res}_{S/d}^{S/n}(F_d(\underline{A})) = (A_{\nu d})_{\nu \in S/n} \rightarrow (A_{\nu n})_{\nu \in S/n} = F_n(\underline{A})$$

is the map with ν -th component $\pi_{\nu d, \nu n} : A_{\nu d} \rightarrow A_{\nu n}$. It follows from Proposition 12 that the maps F_p equip $\underline{W}(\underline{A})$ with a commuting family of Frobenius lifts as in the Dwork lemma. The commutation property of the F_p follows from the known commutation properties of the universal polynomials defining the Witt vector Frobenius morphisms. The surjective ring homomorphisms

$$\underline{W}(\underline{A})_n = W_{S/n}(F_n(\underline{A})) \xrightarrow{\text{res}} F_n(\underline{A})_1 = A_n$$

are compatible with the transition maps $\pi_{d,n}$ for $d \mid n, n \in S$. Hence we obtain a map of inductive systems of rings

$$\text{res} : \underline{W}(\underline{A}) \rightarrow \underline{A} .$$

The ghost maps

$$\mathcal{G}_{S/n} : W_{S/n}(F_n(\underline{A})) \rightarrow \prod_{k \in S/n} A_{kn}$$

combine into a morphism of inductive systems of rings

$$\underline{\mathcal{G}} : \underline{W}(\underline{A}) \rightarrow (\prod_{k \in S/n} A_{kn})_{n \in S}$$

such that $\text{res} \circ \underline{\mathcal{G}} = \text{res}$.

Universal property of \underline{W} In the situation of the Dwork lemma, assume in addition that A_n has no n -torsion for each $n \in S$. Moreover we suppose that ϕ_p commutes with ϕ_l for all primes $p, l \in S$. This means that for every $n \in S$ with $p \mid n$ and $l \mid n$ the following diagram commutes:

$$\begin{array}{ccc} A_{n/pl} & \xrightarrow{\phi_p} & A_{n/l} \\ \phi_l \downarrow & & \downarrow \phi_l \\ A_{n/p} & \xrightarrow{\phi_p} & A_n . \end{array}$$

Then there is a unique morphism $\lambda : \underline{A} \rightarrow \underline{W}(\underline{A})$ of inductive systems of rings with the following properties:

- a) $\text{res} \circ \lambda = \text{id}_{\underline{A}}$
- b) λ commutes with the Frobenius lifts on \underline{A} and $\underline{W}(\underline{A})$, i.e. the diagrams

$$\begin{array}{ccc} A_{n/p} & \xrightarrow{\lambda_{n/p}} & \underline{W}(\underline{A})_{n/p} \\ \phi_p \downarrow & & \downarrow F_p \\ A_n & \xrightarrow{\lambda_n} & \underline{W}(\underline{A})_n \end{array}$$

commute for all $p \mid n, n \in S$.

Proof. We first assume that λ satisfying the desired properties exists and determine its form. According to the Dwork lemma and Proposition 11, via the injective ghost map we have isomorphisms for all $n \in S$

$$W(\underline{A})_n = W_{S/n}(F_n(\underline{A})) \cong \{(x_k) \in \prod_{k \in S/n} A_{nk} \mid \phi_l(x_{k/l}) \equiv x_k \pmod{l^{v_l(k)} A_{nk}} \text{ for } l \mid k, k \in S/n\},$$

where l denotes prime numbers. In the following proof, we will view this isomorphism as an identification.

For $m = p_1^{\nu_1} \cdots p_r^{\nu_r}$ dividing $n \in S$ we set

$$\phi_m := \phi_{p_1}^{\nu_1} \circ \cdots \circ \phi_{p_r}^{\nu_r} : A_{n/m} \rightarrow A_n.$$

By assumption, this is independent of the ordering of the primes p_1, \dots, p_r dividing m . For the W -Frobenius morphisms the corresponding formula

$$F_m = F_{p_1}^{\nu_1} \circ \cdots \circ F_{p_r}^{\nu_r} : W(\underline{A})_{n/m} \rightarrow W(\underline{A})_n$$

holds and we therefore have a commutative diagram for $m \mid n, n \in S$

$$\begin{array}{ccc} A_{n/m} & \xrightarrow{\lambda_{n/m}} & W(\underline{A})_{n/m} \\ \phi_m \downarrow & & \downarrow F_m \\ A_n & \xrightarrow{\lambda_n} & W(\underline{A})_n. \end{array}$$

In the representation of $W(\underline{A})_{n/m}$ and $W(\underline{A})_n$ on the ghost side via the Dwork lemma, the map F_m is given by $F_m((x_k)_{k \in S/(n/m)}) = (x_{km})_{k \in S/n}$. For $a \in A_{n/m}$ consider the relation

$$\lambda_n(\phi_m(a)) = F_m(\lambda_{n/m}(a)).$$

The property $\text{res} \circ \lambda = \text{id}$ implies that $\lambda_n(b)_1 = b$ for all $b \in A_n$. Hence

$$\phi_m(a) = \lambda_n(\phi_m(a))_1 = \lambda_{n/m}(a)_m.$$

Setting $\nu = n/m$, it follows that for all $\nu \in S, m \in S/\nu$ and $a \in A_\nu$, we have $\lambda_\nu(a)_m = \phi_m(a)$, i.e. $\lambda_\nu(a) = (\phi_m(a))_{m \in S/\nu}$. Thus we have seen that a map $\lambda : \underline{A} \rightarrow W(\underline{A})$ with properties a) and b) must have the form $\lambda = (\lambda_n)_{n \in S}$ where $\lambda_n : A_n \rightarrow W(\underline{A})_n = W_{S/n}(F_n \underline{A})$ is given by $\lambda_n(a) = (\phi_k(a))_{k \in S/n}$. On the other hand, setting $x_k = \phi_k(a)$ for $a \in A_n$ we have

$$\phi_l(x_{k/l}) = \phi_l \phi_{k/l}(a) = \phi_k(a) = x_k.$$

Hence $\lambda_n(a) = (\phi_k(a))_{k \in S/n}$ is indeed an element of $W(\underline{A})_n$ in the Dwork lemma description above. Clearly λ_n so defined is a ring homomorphism with

$$\lambda_n(a)_1 = \phi_1(a) = a \text{ i.e. } \text{res} \circ \lambda = \text{id}.$$

The maps λ_n are compatible with the transition maps of \underline{A} and $W(\underline{A})$ because the ϕ_p are compatible with transition maps by definition. Finally, we have

$$\lambda_n(\phi_p(a)) = (\phi_k(\phi_p(a)))_{k \in S/n} = (\phi_{kp}(a))_{k \in S/n} = F_p((\phi_k(a)))_{k \in S/(n/p)}$$

i.e. $\lambda_n \circ \phi_p = F_p \circ \lambda_{n/p}$. □

Corollary 13 (Universal property of \tilde{W} over S). *Fix a divisor stable set S and consider inductive systems of rings $\underline{A} = (A_n)_{n \in S}$ and $\underline{B} = (B_n)_{n \in S}$ with the following properties:*

- i) A_n and B_n have no n -torsion for all $n \in S$.
- ii) \underline{A} is equipped with commuting Frobenius lifts.

Then for any morphism $\alpha : \underline{A} \rightarrow \underline{B}$ of ind-rings there is a unique morphism $\beta : \underline{A} \rightarrow \tilde{W}(\underline{B})$ of ind-sets commuting with the Frobenius maps such that the diagram

$$\begin{array}{ccc} & \tilde{W}(\underline{B}) & \\ \beta \nearrow & & \searrow \text{res} \\ \underline{A} & \xrightarrow{\alpha} & \underline{B} \end{array}$$

commutes. The morphism β is a morphism of ind-rings.

Proof. The existence follows from the Catier–Dieudonné lemma by setting $\beta = \tilde{W}(\alpha) \circ \lambda$ and looking at the commutative diagram

$$\begin{array}{ccccc} \underline{A} & \xrightarrow{\lambda} & \tilde{W}(\underline{A}) & \xrightarrow{\tilde{W}(\alpha)} & \tilde{W}(\underline{B}) \\ & \searrow & \downarrow \text{res} & & \downarrow \text{res} \\ & & \underline{A} & \xrightarrow{\alpha} & \underline{B} \end{array}$$

For the uniqueness, assume that we are given another lift (of ind-sets) $\beta' : \underline{A} \rightarrow \tilde{W}(\underline{B})$ of α commuting with the Frobenius maps. Since the B_n have no n -torsion the ghost map $\tilde{W}(\underline{B}) \rightarrow (\prod_{k \in S/n} B_{kn})_{n \in S}$ has injective components.

Hence it suffices to show that a map γ making the following diagram commutative is uniquely determined if it commutes with the Frobenius maps

$$\begin{array}{ccc} & \left(\prod_{k \in S/n} B_{kn} \right)_{n \in S} & \\ \gamma \nearrow & \downarrow \pi & \\ \underline{A} & \xrightarrow{\alpha} & \underline{B} \end{array}$$

Here $\pi = (\pi_n)_{n \in S}$ where $\pi_n : \prod_{k \in S/n} B_{kn} \rightarrow B_n$ maps $(x_k)_{k \in S/n}$ to x_1 . Moreover, commutation with Frobenius maps means that all diagrams

$$\begin{array}{ccc} A_{n/p} & \xrightarrow{\gamma_{n/p}} & \prod_{k \in S/(n/p)} B_{kn/p} \\ \phi_p \downarrow & & \downarrow F_p \\ A_n & \xrightarrow{\gamma_n} & \prod_{k \in S/n} B_{kn} \end{array}$$

for all prime numbers $p \mid n, n \in S$ are commutative. Here

$$F_p((x_k)_{k \in S/(n/p)}) = (x_{pk})_{k \in S/n}.$$

It follows that defining $\phi_m = \phi_{p_1}^{\nu_1} \circ \dots \circ \phi_{p_r}^{\nu_r}$ for $m = p_1^{\nu_1} \cdots p_r^{\nu_r}$ as before, the following diagrams for $m \mid n, n \in S$ commute:

$$\begin{array}{ccc} A_{n/m} & \xrightarrow{\gamma_{n/m}} & \prod_{k \in S/(n/m)} B_{kn/m} \\ \phi_m \downarrow & & \downarrow F_m \\ A_n & \xrightarrow{\gamma_n} & \prod_{k \in S/n} B_{kn} . \end{array}$$

Here $F_m((x_k)_{k \in S/(n/m)}) = (x_{mk})_{k \in S/n}$. The property $\pi_n \circ \gamma_n = \alpha_n$ implies that for $a \in A_{n/m}$ we have

$$\alpha_n(\phi_m(a)) = \pi_n(\gamma_n(\phi_m(a))) = \pi_n(F_m(\gamma_{n/m}(a))) = \gamma_{n/m}(a)_m .$$

Setting $\nu = n/m$ it follows that for all $\nu \in S, m \in S/\nu$ and $a \in A_\nu$ we have

$$\gamma_\nu(a)_m = \alpha_{m\nu}(\phi_m(a)) \quad \text{in } B_{m\nu}$$

or by replacing the index m by the letter k :

$$\gamma_\nu(a) = (\alpha_{k\nu}(\phi_k(a)))_{k \in S/\nu} \quad \text{in } \prod_{k \in S/\nu} B_{k\nu} .$$

□

Remark Consider the map $\beta = (\beta_n) : \underline{A} \rightarrow \underline{W}(\underline{B})$ where

$$\beta_n : A_n \rightarrow \underline{W}(\underline{B})_n = W_{S/n}(F_n(\underline{B})) .$$

In the proof above we have seen that the composition of β_n with the ghost map

$$\mathcal{G} : W_{S/n}(F_n(\underline{B})) \rightarrow \prod_{k \in S/n} B_{kn}$$

is given as follows: For $a \in A_n$ we have the formula:

$$\mathcal{G}(\beta_n(a)) = (\alpha_{kn}(\phi_k(a)))_{k \in S/n} .$$

REFERENCES

- [A] Roland Auer. A functorial property of nested Witt vectors. *J. Algebra*, 252(2):293–299, 2002.
- [CD] Joachim Cuntz and Christopher Deninger. Witt vector rings and the relative de Rham Witt complex. *J. Algebra*, 440:545–593, 2015. With an appendix by Umberto Zannier.
- [DG] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice it Corps de classes local par Michiel Hazewinkel.
- [H] Lars Hesselholt. The big de Rham-Witt complex. *Acta Math.*, 214(1):135–207, 2015.
- [L] Cristian Lenart. Formal group-theoretic generalizations of the necklace algebra, including a q -deformation. *J. Algebra*, 199(2):703–732, 1998.
- [O1] Young-Tak Oh. Nested Witt vectors and their q -deformation. *J. Algebra*, 309(2):683–710, 2007.
- [O2] Young-Tak Oh. Generalizing Witt vector construction. arXiv:1211.3508.